Generalized synchronization of complex networks

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We consider generalized synchronization of complex networks, which are unidirectionally coupled in the drive-response configuration. The drive network consists of linearly and diffusively coupled identical chaotic systems. By choosing suitable driving signals, we can construct the response network to generally synchronize the drive network in a predefined functional relationship. This extends both generalized synchronization of chaotic systems and synchronization inside a network. Theoretical analysis and numerical simulations fully verify our main results.

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In the past decades, an increasing interest has been focused on complex networks [1,2]. Recently, one of the most interesting topics is to study network synchronization [3–7]. Network synchronization can be considered in two ways. (i) Synchronization arises inside a network composed of coupled dynamical systems. Complete synchronization and phase synchronization are two typical types of synchronization phenomena. (ii) Synchronization occurs between two coupled complex networks in the drive-response configuration [8,9]. In this case, one network is the drive network, and the other network is the response network. The driving signal should be chosen suitably such that the drive and response networks are synchronized.

Generally speaking, synchronization of chaotic systems in the drive-response configuration is one special kind of synchronization of networks [10–13]. As far as we know, except for complete synchronization, there exists another well-known phenomenon, namely, generalized synchronization [12,13]. It implies that there exists a functional relationship between the drive and response systems.

Recently, researchers began to study synchronization of networks. Li *et al.* considered synchronization of two unidirectionally coupled networks by the control strategy [8]. Complete synchronization can be ensured if controllers are applied to the response network. Even the drive network is uncertain, Yu *et al.* utilized the adaptive filtering strategy to construct the response network (to estimate the unknown drive network), and two networks can be also completely synchronized [9]. However, the above methods can only ensure complete synchronization, and cannot realize generalized synchronization.

Compared with complete synchronization, generalized synchronization leads to richer behavior. Though the auxiliary system approach developed for chaos synchronization can be directly applied to analyze whether there exists the phenomenon of generalized synchronization of networks [13], the auxiliary system approach is only one sufficient condition. It cannot provide us the detailed functional relationship between the drive and response networks.

In this Brief Report, we also consider generalized synchronization of networks in the drive-response configuration. We first use a simple chaotic system, namely, Genesio-Tesi system, to construct the drive network. By choosing special driving signals, we then construct the response network to generally synchronize the drive network in a predefined functional relationship. At last, we extend our main results using the the Lie derivative operator.

Suppose that the drive network consists of N linearly and diffusively coupled identical nodes, with each node being an n-dimensional chaotic system, in the following form

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1,j\neq i}^{N} G_{ij} L(y_j - y_i) \tag{1}$$

for $1 \le i \le N$, where $x_i=(x_{i1},\ldots,x_{in})^T \in \mathbb{R}^n$ is the state, $y_i \in \mathbb{R}$ is the scalar output variable, $L \in \mathbb{R}^n$ is the inner coupling matrix, $f:\mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear vector-valued function, and σ is the global coupling strength. Matrix $G=(G_{ij}) \in \mathbb{R}^{N\times N}$ represents the network topology. G_{ij} is defined as follows: If there is a connection between nodes i, j, then $G_{ij}=G_{ji}=1$; otherwise $G_{ij}=G_{ji}=0$, and the diagonal elements are $G_{ii}=-\sum_{j=1,j\neq i}^{N} G_{ij}$.

Here a simple Genesio-Tesi system represents the node dynamics: $\dot{x}_i=f(x_i)$, given by

$$\dot{x}_{i1} = x_{i2}, \dot{x}_{i2} = x_{i3}, \dot{x}_{i3} = -cx_{i1} - bx_{i2} - ax_{i3} + (x_{i1})^2 \tag{2}$$

for $1 \le i \le N$, where a,b,c are positive parameters. When $a=0.44$, $b=1.1$, $c=1$, Genesio-Tesi system behaves chaotically [14]. Genesio-Tesi system is first proposed in Ref. [14]. It is used to illustrate the novel harmonic balance method, which is effective to analyze chaotic dynamics in nonlinear systems. In this Brief Report, we first use Genesio-Tesi system to explain our approach. Then we extend our approach to other chaotic systems.

In order to realize generalized synchronization of networks, we assume that (i) the driving signal is chosen to be $y_i=x_{i2}+wx_{i1}$ with a positive parameter w; (ii) $L=[001]^T$; (iii) there exists no isolate cluster in the drive network [Eq. (1)], and G is symmetrical and irreducible; (iv) the drive and response networks have the same topology and the same labels of nodes; and (v) all nodes in the drive network [Eq. (1)] are chaotic, which ensures that the driving signal y_i is bounded.

In the following, by the driving signal $y_i=x_{i2}+wx_{i1}$, we can easily construct the response network, and analyze the condition for generalized synchronization of the drive and response networks. We first transform Genesio-Tesi system [Eq. (2)] using the driving signal y_i. From the viewpoint of control, y_i can be regarded as an input of equation $\dot{x}_{i1}+wx_{i1}=y_i$ [12]. Thus

$$x_{i1}=e^{-wt}x_{i1}(0)+e^{-wt}\int_0^t e^{w\tau}y_i(\tau)d\tau$$

Differentiating x_{i1} for 1, 2 and 3 times, together with Genesio-Tesi system [Eq. (2)], we get

$$\ddot{y}_i+\beta_1\dot{y}_i+\beta_0 y_i+e^{-wt}\left(\int_0^t e^{w\tau}y_i(\tau)d\tau+x_{i1}(0)\right)[\Lambda-x_{i1}]=0,$$

where $\Lambda=(-w)^3+a(-w)^2+b(-w)+c$, $\beta_1=(-w)+a$, and $\beta_0=(-w)^2+a(-w)+b$. Let $\eta_i=e^{-wt}\int_0^t e^{w\tau}y_i(\tau)d\tau$. So the above equation becomes [12]

$$\ddot{y}_i+\beta_1\dot{y}_i+\beta_0 y_i+\eta_i\Lambda-\eta_i^2=O_i(e^{-wt}),$$

where $O_i(e^{-wt})=e^{-2wt}x_{i1}^2(0)+2e^{-2wt}x_{i1}(0)\int_0^t e^{w\tau}y_i(\tau)d\tau-e^{-wt}x_{i1}(0)\Lambda$.

Owning to the boundedness of the driving signal y_i, we have $\lim_{t\to\infty}[O_i(e^{-wt})]=0$. Hence we approximately get

$$\ddot{y}_i+\beta_1\dot{y}_i+\beta_0 y_i+\eta_i\Lambda-\eta_i^2=0$$

From $\dot{\eta}_i=-w\eta_i+y_i$, Genesio-Tesi system [Eq. (2)] is transformed into the dynamics: $\dot{\bar{x}}_i=g(\bar{x}_i)$ where $\bar{x}_i=(y_i,\dot{y}_i,\eta_i)^T$ [12]

$$\dot{\bar{x}}_{i1}=\bar{x}_{i2},\dot{\bar{x}}_{i2}=-\beta_0\bar{x}_{i1}-\beta_1\bar{x}_{i2}-\Lambda\bar{x}_{i3}+(\bar{x}_{i3})^2,$$

$$\dot{\bar{x}}_{i3}=-w\bar{x}_{i3}+y_i. \qquad (3)$$

The parameter w is chosen by (i) $w\in(0,a)$, if $a^2-b<0$; (ii) $w\in(0,a/2)\cup(a/2,a)$, if $a^2-4b=0$; (iii) $w\in(0,(a-\sqrt{a^2-4b})/2)\cup(a+\sqrt{a^2-4b}/2,a)$, if $a^2-4b>0$. Hence we can choose w such that β_1 and β_0 are positive.

In this Brief Report, the response network is constructed as follows:

$$\dot{\bar{x}}_i=g(\bar{x}_i)+\sigma\sum_{j=1,j\neq i}^{N}G_{ij}\bar{L}(\bar{x}_{j1}-\bar{x}_{i1}), \qquad (4)$$

where $\bar{L}=[010]^T$. Now we show that the drive network [Eq. (1)] and the response network [Eq. (4)] are generally synchronized in the sense that $\lim_{t\to\infty}(x_{i2}+wx_{i1}-\bar{x}_{i1})=0$ for $1\leq i\leq N$.

The detailed dynamics the drive and response network are described by

$$\dot{x}_{i1}=x_{i2},$$

$$\dot{x}_{i2}=x_{i3},$$

$$\dot{x}_{i3}=-cx_{i1}-bx_{i2}-ax_{i3}+(x_{i1})^2+\sigma\sum_{j=1,j\neq i}^{N}G_{ij}(y_j-y_i) \qquad (5)$$

and

$$\dot{\bar{x}}_{i1}=\bar{x}_{i2},$$

$$\dot{\bar{x}}_{i2}=-\beta_0\bar{x}_{i1}-\beta_1\bar{x}_{i2}-\Lambda\bar{x}_{i3}+(\bar{x}_{i3})^2+\sigma\sum_{j=1,j\neq i}^{N}G_{ij}(\bar{x}_{j1}-\bar{x}_{i1}),$$

$$\dot{\bar{x}}_{i3}=-w\bar{x}_{i3}+y_i, \qquad (6)$$

where the driving signal y_i can be directly injected into the response network [Eq. (6)].

Define the error $\varepsilon_i=y_i-\bar{x}_{i1}=x_{i2}+wx_{i1}-\bar{x}_{i1}$. Hence we get

$$\ddot{\varepsilon}_i+\beta_1\varepsilon_i+\beta_0\varepsilon_i-\sigma\sum_{j=1,j\neq i}^{N}G_{ij}(\varepsilon_j-\varepsilon_i)=\bar{O}_i(e^{-wt})$$

where $\bar{O}_i(e^{-wt})=e^{-2wt}x_{i1}^2(0)+2e^{-2wt}x_{i1}(0)\int_0^t e^{w\tau}y_i(\tau)\times d\tau-e^{-wt}x_{i1}(0)\Lambda-e^{-2wt}\bar{x}_{i3}^2(0)-2e^{-2wt}\bar{x}_{i3}(0)\int_0^t e^{w\tau}y_i(\tau)d\tau+e^{-wt}\bar{x}_{i3}(0)\Lambda$. Note that $\bar{O}_i(e^{-wt})$ approaches zero as time t tends to infinity. Hence the stability of the above equation is equivalent to the stability of the following equation:

$$\ddot{\varepsilon}_i+\beta_1\dot{\varepsilon}_i+\beta_0\varepsilon_i-\sigma\sum_{j=1,j\neq i}^{N}G_{ij}(\varepsilon_j-\varepsilon_i)=0 \qquad (7)$$

for $1\leq i\leq N$.

Obviously, Eq. (7) can be transformed into

$$\begin{bmatrix}\dot{e}_{i1}\\ \dot{e}_{i2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\beta_0 & -\beta_1\end{bmatrix}\begin{bmatrix}e_{i1}\\ e_{i2}\end{bmatrix}+\sigma\sum_{j=1}^{N}G_{ij}\Gamma\begin{bmatrix}e_{j1}\\ e_{j2}\end{bmatrix} \qquad (8)$$

for $1\leq i\leq N$, where $e_{i1}=\varepsilon_i$, $e_{i2}=\dot{\varepsilon}_i$, and

$$\Gamma=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Note that there exists no isolate cluster in the drive and response networks, and G is symmetrical and irreducible. From Refs. [3,4,6], the stability of Eq. (8) can be transformed into the following $N-1$ subsystems:

$$\begin{bmatrix}\dot{\mu}_{i1}\\ \dot{\mu}_{i2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\beta_0+\sigma\lambda_i & -\beta_1\end{bmatrix}\begin{bmatrix}\mu_{i1}\\ \mu_{i2}\end{bmatrix}, \qquad (9)$$

where λ_i are eigenvalues of G, and $0=\lambda_1>\lambda_2\geq\cdots\geq\lambda_N$. Hence the drive network [Eq. (5)] and the response network [Eq. (6)] are generally synchronized (namely, the limit $\lim_{t\to\infty}\varepsilon_i(t)=0$ for $1\leq i\leq N$) if σ and λ_i make the polynomial

$$\lambda(s)=\lambda^2+\beta_1\lambda+(\beta_0-\sigma\lambda_i) \qquad (10)$$

be Hurwitz stable. The Hurwitz stability means that $\lambda(s)=0$ has two roots $\bar{\lambda}_1$ and $\bar{\lambda}_2$ with $\mathrm{Re}(\bar{\lambda}_1)<0$ and $\mathrm{Re}(\bar{\lambda}_2)<0$.

Since eigenvalues $\lambda_i\leq 0$, $\sigma\geq 0$, $\beta_0>0$, and $\beta_1>0$, the drive network [Eq. (5)] and the response network [Eq. (6)] can be generally synchronized if σ and nonzero λ_i satisfy

$$\beta_0-\sigma\lambda_i>0. \qquad (11)$$

Thought condition (11) holds for arbitrary value σ, the value σ should not be large. This is because states of the drive network [Eq. (5)] should be bounded, which further ensures the driving signal is bounded. One case is to require all nodes in the drive network [Eq. (5)] are chaotic. Accordingly, the coupling strength σ should be sufficiently small.

From Eq. (2), Genesio-Tesi system is somewhat special. This may restrict the application of the proposed approach. Fortunately, under certain conditions, many chaotic systems can be transformed into equation like Eq. (2). Consider chaotic systems given by

$$\dot{x}(t) = F[x(t)], s(t) = h[x(t)],$$

where $F:\mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth nonlinear vector-valued function, $x \in \mathbb{R}^n$ is the state and $s \in \mathbb{R}$ is the scalar output signal. From the scalar signal s and its derivatives of successively higher order, we get the following state:

$$Z = [s(t), \dot{s}(t), \ldots, s^{(m-1)}(t)]^T = [h(x), L_F(h(x)), \ldots, L_F^{m-1}h(x)]^T = H(x),$$

where L denotes the Lie derivative operator [15,16], that is, $L_F^j h(x) = \sum_{i=1}^n \frac{\partial(L_F^{j-1}h)}{\partial x_i} F_i(x)$. Further, as long as m is sufficiently large (for example, $m > 2n$), H is an embedding and $\partial H/\partial x$ is of full rank [16]. It is easy to show that the state Z satisfies [17]

$$\dot{z}_1 = z_2, \dot{z}_2 = z_3, \cdots, \dot{z}_{m-1} = z_m, \dot{z}_m = \varphi(z), \tag{12}$$

where $\varphi(z) = L_F^m h(x) = L_F^m(H^{-1}(Z))$. Similar to the above analysis, we can choose some parameters $a_1, \ldots, a_m > 0$ such that $\dot{z}_m = -a_1 z_1 - \cdots - a_m z_m + \bar{\varphi}(z)$ and $\bar{\varphi}(z) = \varphi(z) + a_1 z_1 + \cdots + a_m z_m$. If we choose $y = z_2 + w z_1$, we can transform Eq. (12). Hence, based on Eq. (12) and its transformation, we can construct the drive network (1) and the response network (4), where $L = [0, 0, \ldots, 0, 1]^T$ and $\bar{L} = [0, \ldots, 0, 1, 0]^T$. If $\bar{\varphi}(z)$ satisfies the Liptchitz condition and parameter w is chosen suitably, we also ensure the stability of $N-1$ subsystems by the Lyapunov stability.

Our analysis and simulation are based on Barabasi-Albert (BA) networks [2]. The drive and response networks have the same topology generated by the standard algorithm. Initially M nodes with labels $i = 1, \ldots, M$ are fully connected. At every time step a new node is introduced to be connected to M existing nodes. The probability that the new node is connected to node i depends on degree k_i, i.e., $\Pi_i = k_i / \sum_j k_j$. In order to measure generalized synchronization, we define the average error as $E(t) = \frac{1}{N}\sum_{i=1}^N |y_i(t) - \bar{x}_{i1}(t)|$.

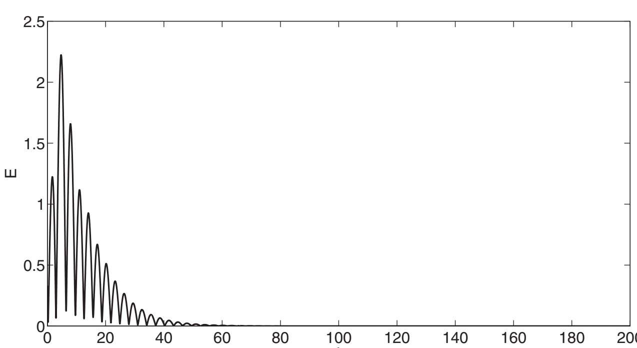

FIG. 1. The average error $E(t)$ versus the time t when there exists no channel disturbance. All estimates are the results of averaging over 100 realizations.

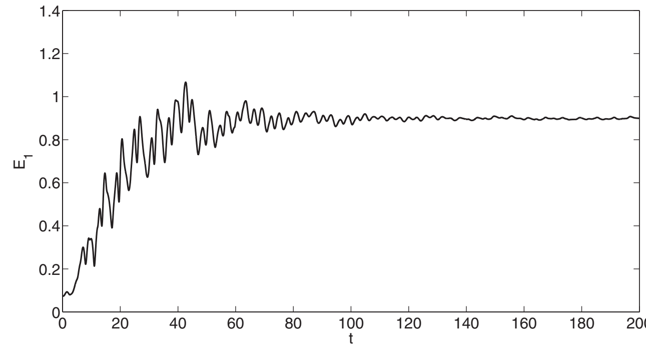

FIG. 2. The average error $E_1(t)$ versus the time t. Here $E_1(t) = \frac{1}{N}\sum_{i=1}^N |x_i(t) - \hat{x}_i(t)|$ and $\hat{x}_i(t) = \frac{1}{N}\sum_{i=1}^N x_i(t)$. All estimates are the results of averaging over 100 realizations.

Throughout our simulations, the number of nodes in networks are $N=500$, and $M=3$. In addition, the parameter $w = 0.2$. Hence parameters $\beta_0 = 1.0520$, $\beta_1 = 0.24$, and $\Lambda = 0.7896$. For BA networks, we can compute eigenvalues of matrix G, and the coupling σ is determined by Eq. (11). After many realizations, we assign the coupling by $\sigma = 0.005$ [satisfying Eq. (11)]. In the drive network [Eq. (5)] and the response network [Eq. (6)], initial states, $x_{i1}(0)$ and $\bar{x}_{i1}(0)$, $x_{i2}(0)$ and $\bar{x}_{i2}(0)$, and $x_{i3}(0)$ and $\bar{x}_{i3}(0)$, are uniformly distributed in $[-0.2, -0.1]$, $[-0.5, -0.4]$, and $[0.8, 0.9]$, respectively. When there exists no channel disturbance, the average error $E(t)$ versus the time t is plotted in Fig. 1. It shows that the drive network and the response network are generally synchronized in the sense that $\lim_{t\rightarrow\infty}(x_{i2} + w x_{i1} - \bar{x}_{i1}) = 0$ for $1 \leq i \leq N$. However, synchronization insider the drive network cannot be ensured (please refer to Fig. 2).

In order to show the effectiveness of our approach, we further consider the robustness to the disturbance in channel (such as channel noise), which often happens during the transition of the driving signal y_i. Suppose that the disturbance $d(t)$ satisfies $|d(t)| < \delta$, where δ is a positive constant. Hence the transmitted signal becomes $y'(t) = y(t) + d(t)$, and it is directly injected into the response network. Here parameters N, M, w, β_0, β_1, and Γ are chosen as above. The disturbance is chosen to be the white noise with $\delta = 0.5$. The average error $E(t)$ versus the time t is plotted in Fig. 3. It shows that the drive network and the response network are almost generally

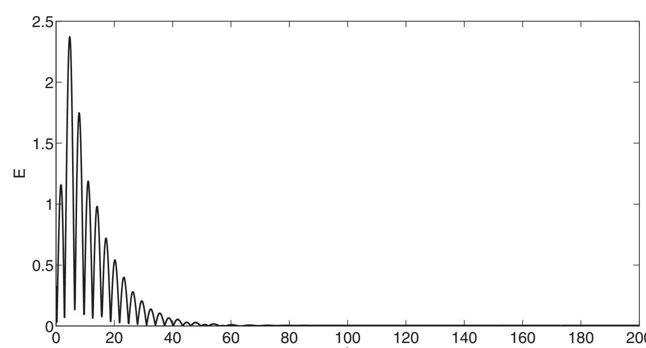

FIG. 3. The average error $E(t)$ versus the time t when there exists channel disturbance. All estimates are the results of averaging over 100 realizations.

synchronized. Therefore, our approach is robust to disturbance in channel.

In this Brief Report, we consider generalized synchronization of complex networks, which are unidirectionally coupled in the drive-response configuration. For the drive network, we choose suitable driving signals and construct the response network, such that the drive and response networks are generally synchronized in a predefined functional relationship. This can be regarded as the extension of both generalized synchronization of chaotic systems and chaos synchronization inside a network.

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